UNIVERSITY OF THE PUNJAB

B.S. in Computer Science / Second Semester – Spring 2023

Roll No. 093297

Subject: Linear Algebra | Paper: MS-153 | Time: 3 Hrs. | Marks: 60

THE ANSWERS MUST BE ATTEMPTED ON THE ANSWER SHEET PROVIDED

Solve the followings: (6x5=30)

Q.1	Show that the matrix $\begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an involutary matrix.
Q.2	Evaluate $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$.
Q.3	Prove that $< u, v >= u_1v_1 - 2u_1v_2 - 2u_2v_1 + 5u_2v_2$ is an inner product on R^2, where $u = (u_1, u_2), v = (v_1, v_2) \in R^2$.
Q.4	Determine whether the set of rational numbers Q is a subspace of the vector space R.
Q.5	Check whether the transformation $T: R^3 \to R^3$ defined by $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_1)$ is linear or nonlinear.
Q.6	Show that AA^t is symmetric for any square matrix A.

Solve the followings. (5x6=30)

Q.7	Use Gauss-Jordan method to solve the following system of equations. $5x_1 - 2x_2 + x_3 = 2$, $3x_1 + 2x_2 + 7x_3 = 3$, $x_1 + x_2 + 3x_3 = 2$
Q.8	Find the inverse of the matrix $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$ by adjoint method.
Q.9	Determine whether or not the vectors $(5,3,-3)$, $(1,0,0)$ and $(0,1,-1)$ are linearly independent or linearly dependent.
Q.10	Find the rank of the matrix $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 15 & 8 & 1 & 12 \\ 11 & 5 & 8 & 6 \\ 12 & 8 & 7 & 10 \end{bmatrix}$.
Q.11	Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$.

Question No.1

Show that the matrix is an involutry matrix.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A \times A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence Proved.

$$A^2 = I$$

Question No. 2

Evaluate $\begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$

Taking Transpose

$$A^t = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \Rightarrow A = -\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

$A = -A$

$A + A = 0$

$2A = 0$

$A = 0$

Question No. 4

Determine whether the set of rational numbers Q is a subspace of the vector space R.

$Q(\mathbb{R})$

$\pi \in \mathbb{R}$

$1 \in Q$

$\pi \cdot 1 = \pi \notin Q$

$\Rightarrow$ $Q(\mathbb{R})$ is not a vector space.

Question No. 6

Show that AA^t is symmetric for any square matrix A.

consider $B = AA^t$

For this purpose we will have to show that

$(B)^t = B \qquad \because B = AA^t$

$B^t = AA^t$

$B^t = (AA^t)^t$

$B^t = (A^t)^t A^t$

$B^t = A A^t$

$B^t = B$

Hence proved AA^t is symmetric.

Question No. 7

Use Gauss-Jordan method to solve the system

$5x_1 - 2x_2 + x_3 = 2$

$3x_1 + 2x_2 + 7x_3 = 3$

$x_1 + x_2 + 3x_3 = 2$

$$\left[\begin{array}{ccc|c} 5 & -2 & 1 & 2 \\ 3 & 2 & 7 & 3 \\ 1 & 1 & 3 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 3 & 2 & 7 & 3 \\ 5 & -2 & 1 & 2 \end{array}\right] \text{ Interchanging } R_1 \,\&\, R_3$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & -1 & -2 & -3 \\ 0 & -7 & -14 & -8 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 5R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & -1 & -2 & -3 \\ 0 & 0 & 0 & 13 \end{array}\right] \quad R_3 - 7R_2$$

System is inconsistent so, it has no solution.

Question No. 8

Find the inverse of matrix

$$A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}$$ by adjoint method.

$$A_{11} = (-1)^2 \begin{vmatrix} -1 & 8 \\ -2 & 7 \end{vmatrix} = (1)(-7+16) = 9$$

$$A_{12} = (-1)^3 \begin{vmatrix} 2 & 8 \\ 5 & 7 \end{vmatrix} = (-1)(14-40) = (-1)(-26) = 26$$

$$A_{13} = (-1)^4 \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = (1)(-4+5) = 1$$

$$A_{21} = (-1)^3 \begin{vmatrix} 4 & 5 \\ -2 & 7 \end{vmatrix} \Rightarrow (-1)(28+10) = -38$$

$$A_{22} = (-1)^4 \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} \Rightarrow (1)(21-25) \Rightarrow -4$$

$$A_{23} = (-1)^5 \begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix} \Rightarrow (-1)(-6-20) \Rightarrow 26$$

$$A_{31} = (-1)^4 \begin{vmatrix} 4 & 5 \\ -1 & 8 \end{vmatrix} \Rightarrow (1)(32+5) = 37$$

$$A_{32} = (-1)^5 \begin{vmatrix} 3 & 5 \\ 2 & 8 \end{vmatrix} \Rightarrow (-1)(24-10) = -14$$

$$A_{33} = (-1)^6 \begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} \Rightarrow (1)(-3-8) = -11$$

$$A = \begin{bmatrix} 9 & 26 & 1 \\ -38 & -4 & 26 \\ 37 & -14 & -11 \end{bmatrix}$$

Taking Transpose

$$\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$$

$$|A| = 3(-7+16) - 4(14-40) + 5(-4+5)$$

$$= 3(9) - 4(-26) + 5(1)$$

$$= 27 + 104 + 5 \Rightarrow 136$$

$$A^{-1} = \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix} \frac{1}{136}$$

$$A^{-1} = \begin{bmatrix} 9/136 & -38/136 & 37/136 \\ 26/136 & -4/136 & -14/136 \\ 1/136 & 26/136 & -11/136 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 9/136 & -19/68 & 37/136 \\ 13/68 & 1/34 & 7/68 \\ 1/136 & 13/68 & 11/136 \end{bmatrix}$$

Question No. 10

Find rank of matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 15 & 8 & 1 & 12 \\ 11 & 5 & 8 & 6 \\ 12 & 8 & 7 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 3 & 5 \\ 15 & 8 & 1 & 12 \\ 11 & 5 & 8 & 6 \\ 12 & 8 & 7 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/2 & 3/2 & 5/2 \\ 15 & 8 & 1 & 12 \\ 11 & 5 & 8 & 6 \\ 12 & 8 & 7 & 10 \end{bmatrix} \quad R_1/2$$

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 5/2 \\ 0 & 1/2 & -43/2 & -51/2 \\ 0 & -1/2 & -17/2 & -43/2 \\ 0 & 2 & -11 & -20 \end{bmatrix} \begin{array}{l} \\ R_2 - 15R_1 \\ R_3 - 11R_1 \\ R_4 - 12R_1 \end{array}$$

$$\begin{bmatrix} 1 & 1/2 & 3/2 & 5/2 \\ 0 & 1 & -43 & -51 \\ 0 & -1/2 & -17/2 & -43/2 \\ 0 & 2 & -11 & -20 \end{bmatrix} \begin{array}{l} \\ R_2 \times 2 \\ \\ \\ \end{array}$$

$$= \begin{bmatrix} 1 & 1/2 & 3/2 & 5/2 \\ 0 & 1 & -43 & -51 \\ 0 & 0 & -30 & -47 \\ 0 & 0 & 75 & 82 \end{bmatrix} \begin{array}{l} \\ \\ R_3 + R_2/2 \\ R_4 - R_2/2 \end{array}$$

$R_1 - R_2/2$

$$= \begin{bmatrix} 1 & 0 & 23 & 28 \\ 0 & 1 & -43 & -51 \\ 0 & 0 & -30 & -47 \\ 0 & 0 & 75 & 82 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 23 & 28 \\ 0 & 1 & -43 & -51 \\ 0 & 0 & 1 & \frac{47}{30} \\ 0 & 0 & 75 & 82 \end{bmatrix} \; R_3/-30$$

$$= \begin{bmatrix} 1 & 0 & 0 & -241/30 \\ 0 & 1 & 0 & 491/30 \\ 0 & 0 & 1 & 47/30 \\ 0 & 0 & 0 & -123 \end{bmatrix} \begin{array}{l} R_1 - 23R_3 \\ R_2 + 43R_3 \\ \\ R_4 - 75R_3 \end{array}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -241/30 \\ 0 & 1 & 0 & 491/30 \\ 0 & 0 & 1 & 47/30 \\ 0 & 0 & 0 & 1 \end{bmatrix} \; R_4/-123$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{l} R_1 + \frac{241}{30} R_4 \\ R_2 - \frac{491}{30} R_4 \\ R_3 - \frac{47}{30} R_4 \\ \\ \end{array}$$

The Rank of matrix is 4.

Question No. 11

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$$tI - A = \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix} - \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} t-3 & -1 & -1 \\ -2 & t-4 & -2 \\ -1 & -1 & t-3 \end{bmatrix}$$

$$|tI - A| = \Delta(t) = t-3 \begin{vmatrix} t-4 & -2 \\ -1 & t-3 \end{vmatrix} + 1 \begin{vmatrix} -2 & -2 \\ -1 & t-3 \end{vmatrix} - 1 \begin{vmatrix} -2 & t-4 \\ -1 & -1 \end{vmatrix}$$

$$= (t-3)((t-4)(t-3)-2) + 1(-2+6-2) - 1(2+t-4)$$

$$= (t-3)(t^2-3t-4t+12-2) + 1(-2t+4) - 1(t-2)$$

$$= (t-3)(t^2-7t+10) - (2t+4 - t+2$$

$$= t^3 - 7t^2 + 10t - 3t^2 + 21t - 30 - 3t + 6$$

$$= t^3 - 10t^2 + 28t - 24$$

$$\Delta(t) = 0$$

$$t^3 - 10t^2 + 28t - 24 = 0$$

CamScanner

Put $t=0$

$-24 \neq 0$

Put $t=1$

$1-10+28-24=0 \Rightarrow -5 \neq 0$

Put $t=2$

$(2)^3 - 10(2)^2 + 28(2) - 24 = 0$

$8 - 40 + 56 - 24 = 0$

$-32 + 32 = 0$

$0 = 0$

So eigen values is 2.

Question No. 9

Determine whether or not the vectors (~~s~~ are linearly independent or linearly dependent.

$V_1 = (5, 3, -3)$

$V_2 = (1, 0, 0)$

$V_3 = (0, 1, -1)$

$$A = \begin{bmatrix} 5 & 1 & 0 \\ 3 & 0 & 1 \\ -3 & 0 & -1 \end{bmatrix}$$

$$5\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ -3 & -1 \end{vmatrix} + 0\begin{vmatrix} 3 & 0 \\ -3 & 0 \end{vmatrix}$$

$\det(A) = 5(0) - 1(-3+3) + 0$

$= 0 - 0 + 0 \quad \det(A) = 0$ So it is linearly dependent.