THE ANSWERS MUST BE ATTEMPTED ON THE ANSWER SHEET PROVIDED

Q.1. Write short note on the following: (6x5=30)

(i) Find all values of k for which the given augmented matrices corresponds to a consistent linear system.

a) $\begin{bmatrix} 1 & k & -1 \\ 4 & 8 & -4 \end{bmatrix}$ b) $\begin{bmatrix} k & 1 & -2 \\ 4 & -1 & 2 \end{bmatrix}$

(ii) Sketch the unit circle in R^2 using the given inner product

$$\langle u, v\rangle = \frac{1}{4}u_1v_1 + \frac{1}{16}u_2v_2$$

(iii) Prove that

$$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$$

(iv) Show that the vector $v = (6,\ 11,\ 6)$ can be written with the linear combination of the following vectors

$$v_1 = (2,1,4), \quad v_2 = (1,-1,3), \quad v_3 = (3,2,5)$$

(v) Define $T: R^3 \to R^3$ by $T(x_1, x_2, x_3) = (-x_3, x_1, x_1 + x_3)$. Find $N(T)$. Is T one-to-one?

(vi) Show that the following matrix is involutery

$$\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

Answer the following questions. (4x7.5=30)

Q.2 Solve the system of linear equations by Gauss Jorden method

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x - 3w &= -3 \end{aligned}$$

Q.3 Find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the following matrix.

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Q.4 Determine whether the vectors are linearly independent or not?

$v_1 = (1,-2,3), v_2 = (5,6,-1), v_3 = (3,2,1)$.

Q.5 Show that the following are sets are the subspaces of their corresponding vector spaces

a) The set of lines passing through origin in R^2 i.e.

$$\{(x,y) \in R^2 \mid y = ax, \quad a \text{ is any real scalar}\}$$

b) The set of planes passing through origin in R^3 i.e.

$$\{(x,y,z) \in R^3 \mid ax + by + cz = 0, \quad a,b,c \text{ are any real scalars}\}$$

Question no. 1

Short Questions:

(i)

$$\begin{bmatrix} 1 & K & -1 \\ 4 & 8 & -4 \end{bmatrix}$$

Sol:

$$\left[\begin{array}{cc|c} 1 & K & -1 \\ 4 & 8 & -4 \end{array}\right] \quad R_2 - 4R_1$$

$$\left[\begin{array}{cc|c} 1 & K & -1 \\ 0 & 8-4K & 0 \end{array}\right]$$

$$8 - 4K = 0$$

$$\cancel{-}4K = \cancel{-}8$$

$$4K = 8$$

$$K = \frac{8}{4}$$

$$\boxed{K = 2}$$

(ii)

$$\left[\begin{array}{cc|c} K & 1 & -2 \\ 4 & -1 & 2 \end{array}\right]$$

Sol:

$$\begin{bmatrix} ak & b & = -2 \\ 4a & -b & = 2 \end{bmatrix} \begin{matrix} \to (i) \\ \to (ii) \end{matrix}$$

$$4(k) - b = 2$$

$$4k = 2+1$$

$$4k = 3$$

$$\boxed{k = \frac{3}{4}}$$

(iii)b.

$$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$$

Sol:

$$\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix} \begin{matrix} R_1(c) \\ R_2(a) \\ R_3(b) \end{matrix}$$

$$\Rightarrow \frac{1}{abc} \begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ b^2-c^2-a^2 & b^2-c^2-a^2 & c^2+a^2 \end{vmatrix} \begin{matrix} C_2 - C_3 \\ C_1 - C_3 \\ \end{matrix}$$

$$\frac{1}{abc} \begin{vmatrix} a^2+b^2-c^2 & 0 & c^2 \\ 0 & b^2+c^2-a^2 & a^2 \\ -2a^2 & -2c^2 & 0 \end{vmatrix} R_3 - (R_1 + R_2)$$

Expanding from R_1:

$$\frac{1}{abc}\left[a^2+b^2-c^2 \left|0 + 2c^2a^2\right| - 0 + c^2 \left|+2a^2 (b^2+c^2-a^2)\right)\right]$$

$$\frac{1}{abc}\left[2c^2a^4 + 2a^2b^2c^2 - 2a^2c^4\right] + c^2\left[2a^2b^2 - 2a^2c^2 - 2a\right]$$

$$\frac{1}{abc}\left[\cancel{2c^2a^4} + 2a^2b^2c^2 - \cancel{2a^2c^4} + 2a^2b^2c^2 + \cancel{2a^2c^4} - 2\cancel{a^4}c^2\right]$$

$$\frac{1}{\cancel{abc}}\left[4a^2b^2c^2\right]$$

$$\boxed{\Delta = 4abc}$$

Aivban.

$V = (6, 11, 6)$

$V_1 = (2, 1, 4)$

$V_2 = (1, -1, 3)$

$V_3 = (3, 2, 5)$

Sol:

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 6 \\ 1 & -1 & 2 & 11 \\ 4 & 3 & 5 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 2 & 1 & 3 & 6 \\ 4 & 3 & 5 & 6 \end{array}\right] \quad R_{12}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 3 & -1 & -16 \\ 0 & 7 & -3 & -38 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 4R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & -1/3 & -16/3 \\ 0 & 7 & -3 & -38 \end{array}\right] \quad \frac{R_2}{3}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 11 \\ 0 & 1 & -1/3 & -16/3 \\ 0 & 0 & -2/3 & -2/3 \end{array}\right] \quad R_3 - 7R_2$$

$$x - y + 2z = 11 \rightarrow (i)$$

$$y - \frac{1}{3}z = \frac{-16}{3} \rightarrow (ii)$$

$$\frac{-2}{3}z = \frac{-2}{3} \rightarrow (iii)$$

From eq. (iii)

$$\cancel{-}\frac{2}{3}z = \cancel{-}\frac{2}{3}$$

$$z = \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{\cancel{2}}$$

$$\boxed{z = 1}$$

Put in eq. (ii)

$$y - \frac{1}{3}(1) = \frac{-16}{3}$$

$$y - \frac{1}{3} = \frac{-16}{3}$$

$$y = \frac{-16}{3} + \frac{1}{3}$$

$$y = \frac{-\cancel{15}}{\cancel{3}} \Rightarrow \boxed{y = -5}$$

Put in eq. (i)

$$x - (-5) + 2(1) = 11$$

$$x + 5 + 2 = 11$$

$$x = -7 + 11$$

$$\boxed{x = 4}$$

$$V = 4V_1 - 5V_2 + V_3$$

Q(vi)

$$\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

Sol.

$A^2 = I$

$$\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ 0+12-12 & 3+9-12 & -3-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence prove the matrix is involutory.

Long Questions

Question no. 2

Solve the system of linear equations by Gauss Jorden method.

$$x - y + 2z - w = -1$$

$$2x + y - 2z - 2w = -2$$

$$-x + 2y - 4z + w = 1$$

$$3x - 3w = -3$$

Sol:

$$\begin{bmatrix} 1 & -1 & 2 & -1 & = -1 \\ 2 & 1 & -2 & -2 & = -2 \\ -1 & 2 & -4 & 1 & = 1 \\ 3 & 0 & 0 & -3 & = -3 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 3R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \end{array}\right] \begin{array}{l} \\ R_2/3 \\ R_3 - R_2 \\ \frac{R_4}{3} \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \; R_{34}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \; R_3 - R_2$$

<u>It has many solution.</u>

<u>Question no.4</u>

$V_1 = (1, -2, 3)$

$V_2 = (5, 6, -1)$

$V_3 = (3, 2, 1)$

sol:

$$\begin{bmatrix} 1 & 5 & 3 \\ -2 & 6 & 2 \\ 3 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ -1 & 3 & 1 \\ 3 & -1 & 1 \end{bmatrix} \quad \frac{R_2}{2}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & 8 & 4 \\ 0 & -16 & -8 \end{bmatrix} \quad \begin{matrix} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & 8 & 4 \\ 0 & -16 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 3 \\ 0 & 8 & 4 \\ 0 & 0 & -4 \end{bmatrix} \quad R_3 + 2R_2$$

so, prove that vectors are linearly independent.

Question no.5

$w = \{(x, y) \in R^2 \mid y = ax$

a is any seal scalas.

Sol:

$U = (1,1) \in W$

$V = (2,2) \in W$

$U = (1,1) = a 1x, 1y$

$V = (2,2) = a 2x, 2y$

Then:

$U+V = (ax, y) + (a2x, 2y)$

$= (1,1) + (2,2)$

$= (3,3) \in W$

Now:

$KV = K(2,2)$

$KV = 2K, 2K$

This is the subspace of vector set w.

Part b.

$w = \{ (x,y,z) \in R^2 / ax+by+cz=0$

(i) Let x, y, z are zero.

$a(0) + b(0) + c(0) = 0$

$0 = 0 \in W$

Now:

$U = (3,3,3) = a3, b3, c3 \in W$

$V = (4,4,4) = a4, b4, c4 \in W$

$U+V = (3,3,3) + (4,4,4)$

$= (7,7,7) \in W$

$= 7a, 7b, 7c$

Then:

$KV = K(4,4,4)$

$= 4K, 4K, 4K \in W$

So this is the subspace of set vector W.

Question no.3

Find eigenvalue:

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

$$\Delta(t) = \det(tI_n - A)$$

$$= t\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix} - \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} t-1 & 3 & -3 \\ -3 & t+5 & -3 \\ -6 & 6 & t-4 \end{bmatrix}$$

Take determinant:

$$= (1-t)(-5-t)(4-t) - (-6)(3) - (-3)(3(4-t) - 3(-5)) + 3$$

$$(3(-6) - (-5-t)6)$$

$$= 1-t\,(t^2+t-2) - (-3)(3t-6) + 3(6t+30)$$

$$= -t^3 - 4t + 4t \Rightarrow \text{for eigenvalues:}$$

$$\det(tI_n - A) = 0 \quad \Rightarrow \quad -t^3 - 4t^2 + 4t = 0$$

Take common (-):

$$-(t^3 + 4t^2 - 4t) = 0 \quad \Rightarrow \quad t^3 + 4t^2 - 4t = 0$$

$$t(t^2 + 4t - 4) = 0 \quad \Rightarrow \quad t = 0$$

$$t^2 + 4t - 4 = 0$$

using quadratic formula:

$$a = 1,\ b = 4,\ c = -4$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-4)}}{2}$$

$$= \frac{-4 \pm \sqrt{16+16}}{2} \Rightarrow \frac{-4 \pm \sqrt{32}}{2}$$

$$= \frac{-4 \pm 4\sqrt{2}}{2}$$

$$= \frac{\cancel{4}(-1 \pm \sqrt{2})}{\cancel{2}}$$

$$x = -2 + \sqrt{2} \quad ; \quad x = -2 - \sqrt{2}$$

So, the eigenvalues are:

$$(0, -2+\sqrt{2}, -2-\sqrt{2})$$

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